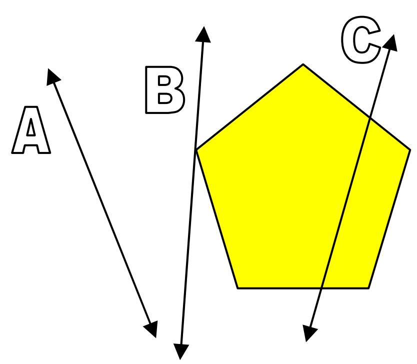
# Introduction to the Gale transform

The Gale transform converts a *d*-dimensional polytope with n vertices into a set of n vectors in *k*-dimensional space where $k = n - d - 1$. This transformation preserves all the combinatorial structure of the polytope, plus more. The transform is particularly useful for low values of $n - d$ but in general it emphasizes the similarities between polytopes with a particular value of $n - d$ as well as the increase in complexity as this difference increases.

## Combinatorics of Polytopes

A hyperplane is called a **supporting hyperplane** for the polytope P, if its intersection with P is non-empty and P is entirely contained in one of the two closed half-spaces determined by the hyperplane.




Line B is a supporting hyperplane for the yellow pentagon.

Lines A and C are not.

The intersection of a polytope with a supporting hyperplane is called a proper **face** of the polytope.

If we allow the empty set and the entire polytope to be included as non-proper faces, then the set of all faces form a partially ordered set under set inclusion. Moreover, this partially ordered set is a particularly nice kind of poset, called a lattice. Two polytopes are called **combinatorially equivalent** (or belong to the same **combinatorial class**) if they have isomorphic face lattices. The information contained in the face lattice is equivalent to knowing which subsets of vertices form a face.

An equivalent definition for face – considered as a subset of vertices – is any set of vertices that take on the value zero for an affine function that has a non-negative value at each vertex of the polytope. (For those unfamiliar, an affine function is simply a linear function plus a constant. The set of zeros for any linear function is a hyperplane through the origin. The set of zeros for an affine function is a general hyperplane.)

## The Gale transformation

Let $\{v_1, v_2, \cdots, v_n\}$ be column vectors representing the coordinates for the vertices of a polytope in d-space, with $n > d+1$. Form the $(d+1)\times n$ matrix $V := \begin{pmatrix} 1 & \cdots & 1 \\ v_1 & \cdots & v_n \end{pmatrix}$. The polytope is full-dimensional if and only if V is full rank. Let G be any $n \times (n-d-1)$ matrix of full rank such that $VG = \vec{0}$. (For example, if $V = \begin{pmatrix} A & B \end{pmatrix}$ with B invertible, we may choose $G = \begin{pmatrix} I_k \\ -B^{-1}A \end{pmatrix}$.) Note that the number of columns of G depends only on the difference between $n$, the number of vertices, and $d$, the dimension.

Matrix multiplication requires that the number of columns of $V$ equals the number of rows of $G$ but the Gale transform will actually associate the i$^{th}$ column of $V$ with the i$^{th}$ row of $G$. Combinatorial properties of the i$^{th}$ vertex will be interpretable as vector properties of the i$^{th}$ row of $G$.

The equation $VG = \vec{0}$ implies that $row(V) \subseteq null(G)$ but a dimension argument, using the fact that V and G are full rank, implies that the two spaces must be equal. (Here $row(V)$ denotes the "row space" of V, i.e. the set of all vectors expressible as $(\vec{r})^T V$, where $(\vec{r})^T$ may be any (d+1)-dim row vector. Also, $null(G)$ refers to the left null space of the matrix G.)

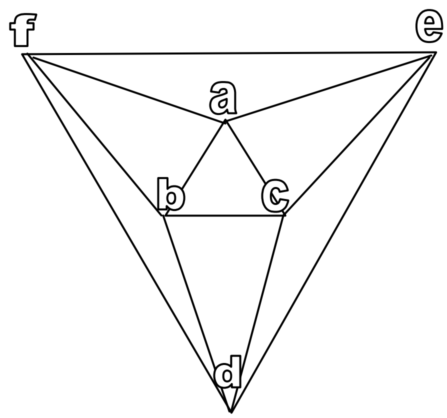
The affine function $f(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d + c$ may be represented as the product $(\vec{r})^T \vec{x}$ where $(\vec{r})^T = \begin{pmatrix} c & a_1 & a_2 & \cdots & a_d \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$.

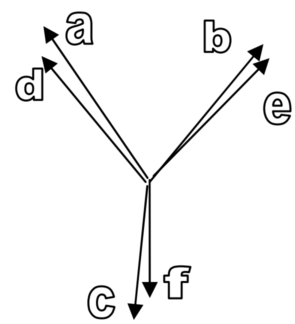
Note: Without the top entry of one, such products could only represent linear functions.

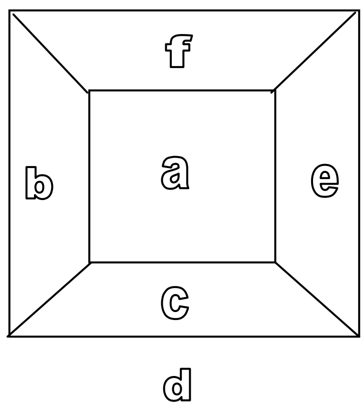
Using $f$ and $\vec{r}$ as above, we get $(\vec{r})^T V = \begin{pmatrix} f(v_1) & f(v_2) & \cdots & f(v_n) \end{pmatrix}$, where $f(v_i)$ is the value of the affine function $f$ at the i$^{th}$ vertex.

The row vector $\begin{pmatrix} f(v_1) & f(v_2) & \cdots & f(v_n) \end{pmatrix}$ is called the **vertex value vector** for function $f$ acting on the polytope. Thus the vectors of $row(V)$ represent all possible vertex value vectors for affine functions on d-space, where the polytope is fixed and each vector in the row space corresponds to a different affine function.

Supporting hyperplanes correspond to vertex value vectors where each entry is non-negative. If we interpret these vectors as elements of $null(G)$ rather than elements of $row(V)$, supporting hyperplanes correspond to linear dependences of the rows of $G$ where each coefficients is non-negative. The vertices that belong to a face of the polytope correspond to the zero coordinates of such a vector.




The upper left diagram represents the octahedron. The upper right diagram gives a set of vectors representing its Gale transform. A set of vertices represents a facet of the octahedron if and only if the vectors corresponding to the complement of that set have a linear dependence using strictly positive coefficients.

The lower left diagram shows the dual of the octahedron (i.e. the cube). A set of facets intersect at a single vertex if and only if the vectors corresponding to the complement of that set have a linear dependence using strictly positive coefficients

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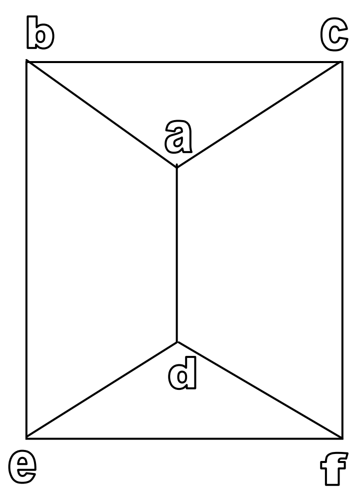


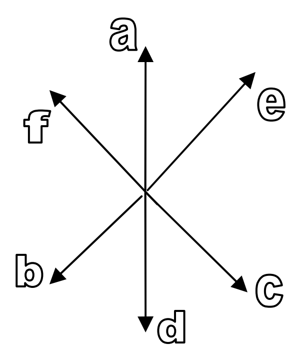


Again, a set of vertices represents a facet of the polytope if and only if the vectors corresponding to the complement of that set have a linear dependence using strictly positive coefficients.

Note that rectangular facets are associated with positive dependences using only two vectors, such as the vectors $\{a, d\}$.

(The dual of this polytope is not shown.)