Magic squares and antimagic graphs

P. Mark Kayll

Department of Mathematical Sciences
University of Montana
Missoula MT 59812-0864, USA
mark.kayll@umontana.edu

Jennifer McNulty

Department of Mathematical Sciences
University of Montana
Missoula MT 59812-0864, USA
jenny.mcnulty@umontana.edu

James Mihalisin

Department of Mathematical Sciences
University of Montana
Missoula MT 59812-0864, USA
mihalisi@mso.umt.edu

20 August 2007

Abstract

An *antimagic labeling* of a graph with m edges is a bijection λ from its edge set to $\{1, 2, \ldots, m\}$ such that the vertex sums are distinct, a vertex sum being the sum of the λ-values on the edges incident with the vertex. An *antimagic* graph is one that admits such a labeling. It was conjectured in [N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *J. Recreational Math.* **21** (1989), 107–115] that every connected graph other than K_2 is antimagic. We verify this conjecture constructively for a class of graphs derived from the complete graphs $K_n = (V, E)$ using a variant of magic squares. In support of our main result, we establish that K_n, with $n \geq 3$, possesses a property stronger than being antimagic: for every function $\omega \colon V \to \mathbb{N}$, there exists a bijection $\lambda \colon E \to \left\{1, 2, \ldots, \binom{n}{2}\right\}$ such that the sums $\omega(v) + \sum_{e \ni v} \lambda(e)$, for $v \in V$, are all different. This 'robust' property of K_n proves useful in establishing that graphs of the form $K_n - F$, for certain $F \subset E$, are antimagic.

Keywords: antimagic, latin square, magic square, RC-magic, transversal system

2000 *MSC*: Primary 05C78 Secondary 05C50, 05B15

1 Introduction

An *antimagic labeling* of a graph $G = (V, E)$ is a bijection $\lambda \colon E \to \{1, \ldots, |E|\}$ such that the *vertex sums* $\lambda(v) := \sum_{e \ni v} \lambda(e)$, for $v \in V$, are all distinct. An *antimagic* graph is one that admits such a labeling. It was conjectured in [7, 8] that every connected graph other than K_2 is antimagic. In recent years, several sophisticated methods have been employed in establishing special cases of this conjecture. Alon *et al.* [3] combined "probabilistic arguments with simple tools from analytic number theory and combinatorial techniques" to verify the conjecture for graphs with minimum degree $\Omega(\log|V|)$. Hefetz [9] used Alon's 'combinatorial Nullstellensatz' (see [2]) to show that any graph of order a power of three and admitting a K_3-factor is antimagic.

We present a few techniques that we have found useful in proving certain graphs are antimagic. Our methods—leaning on properties of magic squares—are at once constructive and more elementary than those of [3, 9]. They yield the following main result.

Theorem 1 *The graph $K_n - T$ is antimagic whenever $T \subset E(K_n)$ is a transversal system of order $\lfloor n/2 \rfloor \geq 3$.*

The definition of 'transversal system' appears immediately preceding Example 3. Since the graphs $K_n - T$ in the statement of Theorem 1 have minimum degree $\Omega(n)$, asymptotically this theorem is a consequence of the result from [3] mentioned in the first paragraph. However, the constant implicit in $\Omega(\log|V|)$ is not made explicit in [3]. We present our constructive proof of Theorem 1 in Section 2 after introducing our expositional necessities.

Notation and terminology

Sets and graphs

Since we frequently sum sets of consecutive integers, it is convenient to abbreviate the sum of the first n nonnegative integers (i.e. elements of $\mathbb{N}$) by $S(n) = n(n+1)/2$; when n is 0 or -1, we view this sum as empty, and in these cases, the formula also gives the desired value $S(0) = S(-1) = 0$. For $m, n \in \mathbb{N}$, with $m \leq n$, we write $S(m,n)$ for $\sum_{i=m}^{n} i = S(n) - S(m-1)$. These conventions imply that

$$S(n-m, n) = n(m+1) - S(m) \text{ for all } m, n \in \mathbb{N} \text{ with } m \leq n. \tag{1}$$

For any omitted graph-theoretic concepts, the reader may consult [5]. If H_1 and H_2 are vertex-disjoint graphs, then $H_1 \oplus H_2$ denotes their union, i.e., the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. The *join* $H_1 \vee H_2$ of H_1 and H_2 is the graph obtained from $H_1 \oplus H_2$ by joining each vertex of H_1 to each vertex of H_2. We use *joining edge* to refer to any member of the edge-set $E(H_1 \vee H_2) \smallsetminus E(H_1 \oplus H_2)$.

Antimagic notions

'Antimagic' is derived from 'magic', a term borrowed for graph theory from 'magic squares'. The last objects date to antiquity (see, e.g., [4] for their background); magic graphs were probably first considered by Sedláček [12]; and a variety of antimagic graph concepts have been studied by numerous authors since the 1990s. See Gallian's extensive survey [6] for background and references on magic and antimagic graph labelings.

For a graph $G = (V, E)$, we call a function $\omega\colon V \to \mathbb{N}$ a *weight function* on V. If such an ω is given, then an *ω-antimagic labeling* of G is a bijection $\lambda\colon E \to \{1, 2, \ldots, |E|\}$ such that the *ω-vertex sums* $\omega(v) + \sum_{e \ni v} \lambda(e)$, for $v \in V$, are all different. An *ω-antimagic* (or, simply, *ω-AM*) graph is one that admits such a labeling; we also use *AM* for 'antimagic'. We call G *shiftably AM* if, in the definition of AM, the range of λ can be replaced by $\{s+1, s+2, \ldots, s+|E|\}$, for any given $s \in \mathbb{N}$, while still achieving the property that all the vertex sums are different. *Shiftably ω-AM* is defined analogously. Finally, we say that G is *robust* if it is shiftably ω-AM for every $\omega\colon V \to \mathbb{N}$. It is worth noting that, using this terminology, the proof in [9] shows that any graph on 3^k vertices that admits a K_3-factor is robust.

Latin and magic squares

Let M be an $n \times n$ matrix. A *transversal* of M consists of a set of n of its entries, with no two residing in the same row or in the same column. Suppose that the entries of M form an arrangement of the integers $\{1, 2, \ldots, n^2\}$. We shall call M *transversal* if each of the sets of entries $\{\ell n + 1, \ell n + 2, \ldots, \ell n + n\}$, for $0 \le \ell \le n-1$, forms a transversal of M. Following [13], we call M *RC-magic (of order n)* in case each row- and column-sum of M is the same. It is easy to see that the common row- and column-sums, the so-called *magic sum*, must be $n(n^2+1)/2$.

Our proof in Section 2 depends in part on RC-magic squares that are also transversal. Such matrices may fail to be magic squares (since we do not insist that their main forward- and back-diagonals sum to the magic sum), but they enjoy the additional property of being transversal. One well-known magic square construction (see, e.g., [1] or [10]) always produces transversal RC-magic squares. Given a pair L, R of orthogonal latin squares on the symbols $\{1, 2, \ldots, n\}$ (and given the all-ones matrix J), form a new matrix M by defining $M = L + n(R - J)$. Our Example 3 below invokes the following result of this construction:

$$\begin{bmatrix} 1 & 10 & 14 & 18 & 22 \\ 23 & 2 & 6 & 15 & 19 \\ 20 & 24 & 3 & 7 & 11 \\ 12 & 16 & 25 & 4 & 8 \\ 9 & 13 & 17 & 21 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}. \tag{2}$$

It is not difficult to verify that M is in general transversal RC-magic of order n or that the following converse holds (*cf.* [1, Theorem 2.22, p. 114]).

Proposition 2 *If M is a transversal RC-magic square of order n, then there exists a pair of orthogonal latin squares of order n.*

Since there exists a pair of orthogonal latin squares of all orders $n \ge 3$, except $n = 6$ (see [1]), the discussion preceding Proposition 2 shows that there exists a transversal RC-magic square $M_n = (m_{ij})$ of each order $n \ge 3$ with $n \ne 6$. Moreover, Proposition 2 shows that no such square of order 6 exists. Thus, our case K_{12} in Theorem 1 requires separate consideration, and for this, we employ the following matrix, whose column-sums are 111 and whose row-sums alternate between 108 and 114:

$$N_6 := \begin{bmatrix} 27 & 35 & 1 & 9 & 17 & 19 \\ 34 & 6 & 8 & 16 & 24 & 26 \\ 5 & 7 & 15 & 23 & 25 & 33 \\ 12 & 14 & 22 & 30 & 32 & 4 \\ 13 & 21 & 29 & 31 & 3 & 11 \\ 20 & 28 & 36 & 2 & 10 & 18 \end{bmatrix}. \tag{3}$$

(We determined the matrix N_6 by mimicking a well-known construction that produces magic squares of all *odd* orders exceeding 1; see, e.g., [11].) Notice that N_6, though not RC-magic, is nevertheless transversal.

We use the matrices N_6 and M_n, for $n \geq 3$ and $n \neq 6$, throughout the remainder of this article.

Transversal systems

Let $M = (m_{ij})$ denote either N_6 or one of our M_n's. Given graphs H_1, H_2 on the respective (disjoint) vertex sets $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_n\}$, we associate a joining edge $\{v_i, w_j\}$ of $H_1 \vee H_2$ with the entry m_{ij} of M. For a nonnegative integer $k \leq n^2$, a *transversal system* T of $H_1 \vee H_2$ (of *size* k and *order* n) consists of a set of joining edges of the form $\{\{v_i, w_j\} : m_{ij} \leq k\}$. Thus, when $k \leq n$, a transversal system T is simply a matching in $H_1 \vee H_2$, and when k is a multiple of n, say $k = rn$, the system T is a collection of r pairwise disjoint perfect matchings from $\{v_1, v_2, \ldots, v_n\}$ to $\{w_1, w_2, \ldots, w_n\}$. We sometimes say that T is *associated* with M; notice that if M has dimension $n \times n$, then there are $n^2 + 1$ transversal systems associated with M. In Theorem 1 (and Corollary 7), we also consider transversal systems T in complete graphs of odd order. In this case, we view K_{2n+1} as $(K_n \vee K_n) \vee K_1$ and restrict T to being a transversal system within the $K_n \vee K_n$ subgraph, as already defined.

Reduced magic square M'_5

0	0	1	5	9
10	0	0	2	6
7	11	0	0	0
0	3	12	0	0
0	0	4	8	0

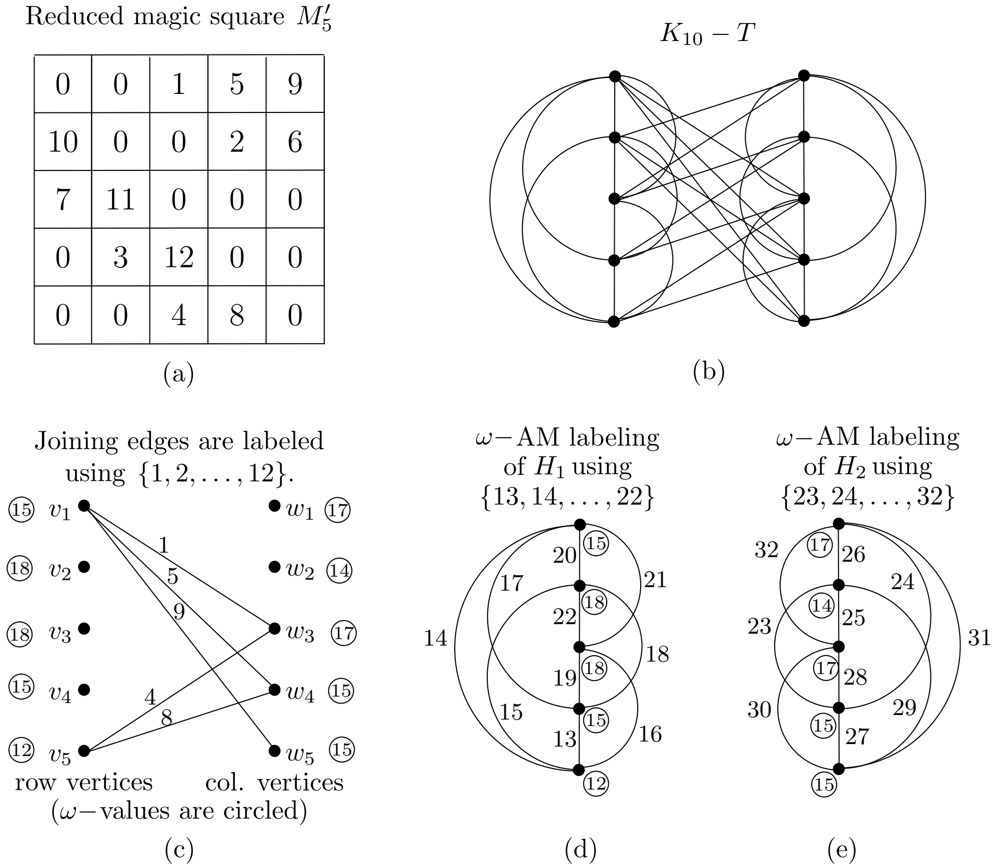

Figure 1: the stages in creating an antimagic labeling of $K_{10} - T = H_1 \vee H_2 - T$

Example 3 To follow the proof of Theorem 6, it will be helpful to consider a concrete example. We construct an antimagic labeling of $K_{10} - T$ for a transversal system T of order 5 and size $k = 13$. For a weight function ω to be specified, we use two (shifted) ω-AM labelings of K_5, together with the transversal RC-magic square $M_5 = (m_{ij})$ given in (2), to which our system T is associated. Define the 'reduced' magic square $M'_5 = (m'_{ij})$ as follows: if $m_{ij} \leq 13$, then $m'_{ij} = 0$; otherwise $m'_{ij} = m_{ij} - 13$ (see Figure 1(a)). Let H_1 and H_2 be complete graphs with respective vertex sets $\{v_1, \ldots, v_5\}$ and $\{w_1, \ldots, w_5\}$, and let $T = \{\{v_i, w_j\} : m'_{ij} = 0\}$. We are thinking of $K_{10} - T$ as $H_1 \vee H_2 - T$ (Figure 1(b)). The edges of this graph fall into three classes: those of H_1, those of H_2, and the joining edges. We use M'_5 to define both a weight function on $V(H_1) \cup V(H_2)$ and the labels of the joining edges. Namely, the weight $\omega(v_i)$ is the ith row-sum of M'_5, the weight $\omega(w_j)$ is the jth column-sum of M'_5, and the label of an edge $\{v_i, w_j\}$ is m'_{ij} (Figure 1(c)). Next, we find an ω-AM labeling of H_1 with labels $\{13, 14, \ldots, 22\}$ and an ω-AM labeling of H_2 with labels $\{23, 24, \ldots, 32\}$ (Figure 1(d,e)). In general, these labelings are delivered by Lemma 4. To see that this labeling of $K_{10} - T$ is antimagic, first note that in defining ω, we arranged for the ω-vertex sums of $H_1 \oplus H_2$ to be the vertex sums of $K_{10} - T$. Since the individual labelings of H_1, H_2 are ω-AM, it follows that the ω-vertex sums within H_1, H_2 are all different. Additionally, the division of the label set ensures that the ω-vertex sums within H_1 are disjoint from those within H_2, and so the vertex sums of $K_{10} - T$ are all different. Finally, the three sets of labels on the three classes of edges partition the label set $\{1, 2, \ldots, 32\}$ required for an AM labeling of $K_{10} - T$.

Tools

The following two results are fundamental tools in proving Theorem 1. The first provides us with our prototypical example of a robust graph. The second allows us to handle the case when n is odd in Theorem 1.

Lemma 4 *The complete graph K_n is robust for $n \geq 3$.*

Proof. Let $s \in \mathbb{N}$ and $\omega \colon V \to \mathbb{N}$ be arbitrary. Index the vertices $v_i \in V$ so that $\omega(v_i) \leq \omega(v_j)$ whenever $i < j$. Order the edges lexicographically and assign the labels $\{s+1, s+2, \ldots, s+\binom{n}{2}\}$ according to this order. Let $\{x_1, x_2, \ldots, x_{n-1}\}$ be the labels assigned to the edges incident to v_i (arranged in increasing order). For $k < i$, the edge $\{v_i, v_k\}$ is labelled x_k. For $k \geq i$, the edge $\{v_i, v_{k+1}\}$ is labelled x_k. Let $\{y_1, y_2, \ldots, y_{n-1}\}$ be the labels assigned to the edges incident to v_j (also arranged in increasing order). By construction, if $i < j$, then, for $1 \leq k \leq n-1$, we must have $x_k \leq y_k$ (with equality only if $j = i+1$ and $k = i$). This implies that the vertex sums are strictly increasing with index, so the ω-vertex sums also must be strictly increasing with index. ∎

The next result was proved in [3]. The proof is included here since it contributes to an explicit construction of an antimagic labeling of each graph considered in Theorem 1.

Lemma 5 *If a graph $G = (V, E)$ contains n vertices and has a vertex of degree $n - 1$, then G is antimagic.*

Proof. Let v be a vertex of degree $n-1$. Using the labels $\{1, 2, \ldots, |E| - n + 1\}$, arbitrarily label all edges not incident to v. For each vertex other than v, calculate the sum of the labels on all incident edges excluding the edge connecting that vertex to v. Index $V \setminus \{v\} =$

$\{v_1, v_2, \ldots, v_{n-1}\}$ so that these (partial) sums are weakly increasing with index. Now for each $j = 1, 2, \ldots, n-1$, assign the label $|E| - (n-1) + j$ to the edge joining v to v_j. Since the partial sums from the arbitrary labeling are weakly increasing with vertex index and the labels of the edges of the form $\{v, v_j\}$ are strictly increasing with j, the vertex sums for $\{v_1, v_2, \ldots, v_{n-1}\}$ must be distinct. Finally, since v has degree at least that of the other vertices and its incident edges are assigned the largest possible labels, its vertex sum strictly dominates the vertex sum of any v_j. ∎

2 Main results

We divide the proof of Theorem 1 according to the parity of the number of vertices. The even case (here denoted $2n$) uses the transversal RC-magic squares M_n (or N_6 if $n = 6$) from the introduction (Theorem 6) while the odd case (here denoted $2n+1$) is an easy consequence of the even case (Corollary 7).

Theorem 6 *If T is a transversal system of order $n \geq 3$, then $K_{2n} - T$ is antimagic.*

If G is an antimagic graph, then adding a vertex to G of degree $|V(G)|$ preserves the antimagic property; this is a special case of Lemma 5. Thus, the following result is immediate.

Corollary 7 *If T is a transversal system of order $n \geq 3$, then $K_{2n+1} - T$ is antimagic.*

The proofs of Theorem 6 and Lemmas 4 and 5 can be used to construct an explicit antimagic labeling of $K_m - T$, for $m \geq 6$, when T is a transversal system of order $n = \lfloor m/2 \rfloor \geq 3$. While a transversal system of order n is defined only for $n \geq 3$, one can ask when $K_m - M$ is antimagic for a matching M. The two preceding results answer this question for $m \geq 6$; we leave it for the reader to verify that $K_m - M$ is antimagic for $m = 3, 4, 5$ and any matching M of K_m.

Proof of Theorem 6

Let H_1 and H_2 be complete graphs on the respective vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$, with $n \geq 3$. Let M_n be a transversal RC-magic square of order $n \neq 6$, and let T be an associated transversal system of size k. As in Example 3, we are thinking of K_{2n} as $H_1 \vee H_2$.

We give an antimagic labeling of $H_1 \vee H_2 - T$ with labels $\{1, 2, \ldots, n^2 - k + 2\binom{n}{2}\}$ so that the joining edges receive the smallest contiguous labels $\{1, \ldots, n^2 - k\}$, the edges of H_1 receive labels $\{n^2 - k + 1, \ldots, n^2 - k + \binom{n}{2}\}$, and the edges of H_2 receive labels $\{n^2 - k + \binom{n}{2} + 1, \ldots, n^2 - k + 2\binom{n}{2}\}$. Moreover, for $i = 1, 2$, the labeling of H_i is a shift of an ω-AM labeling of K_n with labels $\{1, 2, \ldots, \binom{n}{2}\}$. Additionally, in our labeling definition, we arrange for each vertex sum of v_i to be less than each vertex sum of w_j (for all i, j).

Consider the matrix M'_n obtained from M_n as follows:

$$m'_{ij} := \begin{cases} 0 & \text{if } m_{ij} \leq k \\ m_{ij} - k & \text{otherwise.} \end{cases}$$

Notice that $0 \leq m'_{ij} \leq n^2 - k$. For an index i with $1 \leq i \leq n$, let R_i denote the ith row-sum of M'_n and C_i the ith column-sum of M'_n. Also, let L denote a line sum, that is the sum of the entries in a row or column. If $r := \lceil k/n \rceil$, then either r or $r-1$ entries in each line of M correspond to edges of T, so each line of M'_n contains r or $r-1$ zero entries.

We can think of the entries of M'_n as $m'_{ij} = m_{ij} - k$ for $m_{ij} > k$ and $m'_{ij} = m_{ij} - k + (k - m_{ij})$ for $m_{ij} \le k$. Thus, a line sum of M'_n may be viewed as $L = n(n^2+1)/2 - kn + c$, where the first term is the magic sum, the term kn results from shifting each line entry by k, and c is a correction term related to the line entries $m_{ij} \le k$. We give upper and lower bounds for L that rely on the transversal property of M_n; i.e., one entry of every transversal $\{\ell n + 1, \ell n + 2, \ldots, \ell n + n\}$, for $0 \le \ell \le n-1$, lies in each line. Thus, the r smallest entries in any line of M_n can be no smaller than the smallest entry in each of the smallest r transversals: $\{1, n+1, 2n+1, \ldots, (r-1)n+1\}$. Similarly, the $r-1$ largest entries in any line of M_n must be no larger than $\{n, 2n, 3n, \ldots, (r-1)n\}$. Thus we obtain the following bounds on the line sum L:

$$\begin{aligned} L &\le n(n^2+1)/2 - kn + [kr - (1 + (n+1) + (2n+1) + \cdots + ((r-1)n+1))] \\ &= n(n^2+1)/2 - kn + kr - (r(r-1)/2)n - r, \end{aligned} \tag{4}$$

and

$$\begin{aligned} L &\ge n(n^2+1)/2 - kn + [k(r-1) - (n + 2n + 3n + \cdots + (r-1)n)] \\ &= n(n^2+1)/2 - kn + kr - (r(r-1)/2)n - k. \end{aligned} \tag{5}$$

Define a weight function ω on $V(H_1)$ by $\omega(v_i) = R_i$, and let λ_1 be an ω-AM labeling of H_1. Similarly, define a weight function ω on $V(H_2)$ by $\omega(w_i) = C_i$, and let λ_2 be an ω-AM labeling of H_2. (The fact that we are using ω for both $\omega\mid_{V(H_1)}$ and $\omega\mid_{V(H_2)}$ should cause no confusion.) We use λ_1, λ_2, and M'_n to define a labeling λ of $H_1 \vee H_2 - T$. Let $s_1 = (n^2 - k)$ and $s_2 = (n^2 - k) + \binom{n}{2}$. Then

$$\lambda(e) := \begin{cases} m'_{ij} & \text{if } e = \{v_i, w_j\} \\ \lambda_1(e) + s_1 & \text{if } e \in E(H_1) \\ \lambda_2(e) + s_2 & \text{otherwise, i.e., if } e \in E(H_2). \end{cases}$$

Since K_n is robust (by Lemma 4), $\lambda(v_i) \ne \lambda(v_j)$ and $\lambda(w_i) \ne \lambda(w_j)$ whenever $i \ne j$. Thus, we need only check that $\lambda(v_i) \ne \lambda(w_j)$ for all i, j. We will in fact show that $\lambda(v_i) < \lambda(w_j)$ for all i, j. From the definitions of λ and ω, it follows that $\lambda(v_i) = \lambda_1(v_i) + \omega(v_i) + s_1(n-1)$, with an analogous identity for w_j. Hence, it suffices to show that

$$\lambda_1(v_i) - \lambda_2(w_j) + \omega(v_i) - \omega(w_j) \quad < \quad (n-1)(s_2 - s_1) = (n-1)\binom{n}{2}. \tag{6}$$

First, we obtain an upper bound on $\lambda_1(v_i)$ and a lower bound on $\lambda_2(w_j)$ by considering the sums of the largest possible labels and the smallest possible labels. These considerations lead to $\lambda_1(v_i) \le S\left(\binom{n}{2} - (n-2), \binom{n}{2}\right)$ and $\lambda_2(w_j) \ge S(1, (n-1))$. Next, identity (1) yields

$$\lambda_1(v_i) - \lambda_2(w_j) \le (n-1)\binom{n}{2} - S(n-2) - S(n-1) = (n-1)\binom{n}{2} - (n-1)^2.$$

Now using the bounds on the line sums of M'_n from (4) and (5), we obtain

$$\omega(v_i) - \omega(w_j) \le k - r.$$

Thus,

$$\lambda_1(v_i) - \lambda_2(w_j) + \omega(v_i) - \omega(w_j) \leq (n-1)\binom{n}{2} + (k-r) - (n-1)^2.$$

If $k \leq n^2 - 2n$, then since $r \geq 1$, we have $(k-r) < (n-1)^2$, and (6) holds. If $n^2 - 2n + 1 \leq k \leq n^2 - n - 1$, then $r = n-1$ and $k < nr$; thus $k - r < nr - r = r(n-1) = (n-1)^2$, and again (6) is satisfied. If $k \geq n^2 - n$, then we repeat the argument above with the 'complementary' transversal RC-magic square D_n (with entries $d_{ij} = (n^2+1) - m_{ij}$) in place of M_n and $n^2 - k$ in the role of k.

When $n = 6$, we use the matrix N_6 defined in (3) to construct an AM labeling of $K_{2n} - T$. Since N_6 is transversal, a similar technique yields an antimagic labeling of $K_{12} - T$. ∎

Concluding remarks

We view Theorem 1 and its proof as a small step toward the Hartfield-Ringel conjecture mentioned in the Introduction. As our title suggests, one of our purposes is to illustrate the connection between the seemingly contrasting properties of being magic (for a square, i.e., a matrix) and being antimagic (for a graph). Perhaps a more important contribution lies with our approach.

Lemma 4 establishes that complete graphs with at least three vertices satisfy a strong AM hypothesis. We expect that most families of graphs are not robust. For example, nonempty stars fail to be robust, even though they are easily seen to be shiftably AM.

The complete graphs K_n play an essential role in our approach for two reasons. Not only do they have the property that every (simple) graph (with $n := |V|$) is of the form $K_n - F$ for some $F \subset E(K_n)$, but also they are robust (when $n \geq 3$). Therefore, the statement of Theorem 1, that puts a transversal system T in place of F, presents a natural strategy for working toward the Hartsfield-Ringel conjecture; namely, determine other edge families F for which $K_n - F$ is antimagic.

References

[1] R.J.R. Abel, A.E. Brouwer, C.J. Colbourn and J.H. Dinitz, Mutually orthogonal latin squares (MOLS), Chap. II.2 in: C.J. Colbourn and J.H. Dinitz, eds., *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, 1996, 111–142.

[2] N. Alon, Combinatorial Nullstellensatz, *Combin. Prob. Comput.* **8** (1999), 7–29.

[3] N. Alon, G. Kaplan, A. Lev, Y. Roditty and R. Yuster, Dense graphs are antimagic, *J. Graph Theory* **47** (2004), 297–309.

[4] W.W. Rouse Ball and H.S.M. Coxeter, *Mathematical Recreations and Essays*, 12th edition, University of Toronto Press, Toronto, 1974.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North Holland, New York, 1976.

[6] J.A. Gallian, A dynamic survey of graph labeling, *Electronic J. Combin.* **14** (2007), #DS6, 180 pp. (electronic).

[7] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *J. Recreational Math.* **21** (1989), 107–115.

[8] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston, 1990 (revised 1994).

[9] D. Hefetz, Anti-magic graphs via the combinatorial Nullstellensatz, *J. Graph Theory* **50** (2005), 263–272.

[10] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

[11] C.A. Pickover, *The Zen of Magic Squares, Circles, and Stars*, Princeton University Press, Princeton, 2002.

[12] J. Sedláček, Problem 27 in: *Theory of Graphs and its Applications* (Proc. Sympos. Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., Prague, 1964, 163–164.

[13] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston, 2001.