VERTEX MAGIC UNIONS OF STARS

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ABSTRACT. A total labelling is called *vertex magic* if the weight at each vertex equals h, the magic constant. This paper provides examples to show that W.D. Wallis's bounds are tight in the edge maximal case for unions of stars. It then explores some other extreme cases involving unions of stars.

INTRODUCTION

For the graph $G = (V, E)$, let $n = |V| + |E|$. A vertex magic total labelling is a one-to-one function $\lambda : E \cup V \to \{1, 2, \ldots, n\}$ such that $\forall x \in V$ we have $\lambda(x) + \sum \lambda(\overline{xy}) = h$ where h is a fixed constant (the magic constant) and the sum runs over all edges $\in E$ which include x. The left hand side of the above expression is called the *weight* of x.

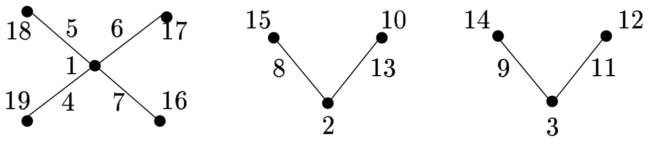

FIGURE 1. A magic labelling for a union of two 2-stars and a 4-star.

The presence of leaves strongly constrains the possible magic constants. If x is the smallest label used on a pendant edge or leaf then $h \leq n + x$. If G is a union of stars, we must have $h = n + x$ and any $i < x$ must label a star center. In general, a graph with l leaves must have $h \leq 2n - 2l + 1$.

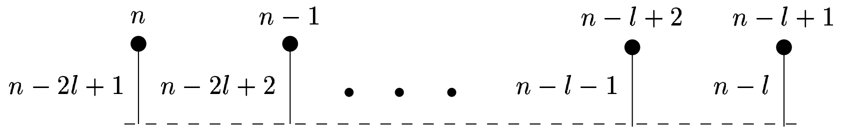

FIGURE 2. This labelling of leaves and pendant edges is necessary to achieve the maximum h.

The author is indebted to the National Science Foundation for its support of this research.

1. UNIONS OF STARS

For integers i, j with $i < j$, define σ_i^j to be $(i+1) + (i+2) + \cdots + j$. Note that $\sigma_0^j = j(j+1)/2$ Given a union of t stars with a total of l leaves, the sum of the weights of the centers is minimized by using the l largest labels on leaves, so we need $\sigma_0^{t+l} \leq t \cdot h$. Combining this with $h \leq 2n - 2l + 1$ yields $l^2 + l(1-2t) - (3t^2 + t) \leq 0$, which implies $l < 3t$. [Wal01]

Since the leaves bound h from above and the centers bound h from below, we might expect that choosing h to be the largest value allowed by the leaves gives the greatest possibility of creating a magic labelling but in fact this choice severely constrains the graphs allowed.

For a union of t stars with l leaves, assume that $h = 2n - 2l + 1$. This implies that the integers $\{1, 2, \ldots, t\}$ must all label centers. Let the s denote the number of leaves for the largest star. By considering the minimum possible weight at its center, we see that:

$$1 + \sigma_t^{t+s} \leq 2n - 2l + 1$$

This may be rewritten as:

$$1 + s \cdot t + s(s+1)/2 \leq 2l + 2t + 1.$$

Using $l < 3t$ gives $s(s+1)/2 \leq (8-2)t$, implying $s \leq 7$. Although a different path was followed, we have arrived at a result very similar to a theorem of Wallis's. [Wal01]

Theorem 1.1. *Suppose G is a union of t stars with l leaves that contains an s-star. Let $n = |V| + |E|$. For G to have a vertex-magic total labelling with magic constant $h = 2n - 2l + 1$,*

$$\begin{array}{lll} \text{if} \quad s = 7 & \text{then} & t \geq 30 \\ \text{if} \quad s = 6 & \text{then} & t \geq 12 \\ \text{if} \quad s = 5 & \text{then} & t \geq 6 \\ \text{if} \quad s = 4 & \text{then} & t \geq 3 \\ \text{if} \quad s = 3 & \text{then} & t \geq 2. \end{array}$$

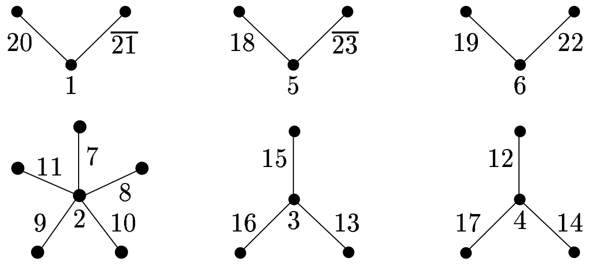

FIGURE 3. Only star centers and edges have been labelled, $\overline{x} = h - x$.

The above labelling will be denoted $\{(h = 47)\{2 : 7, 8, 9, 10, 11\}, \{4 : 12, 14, 17\}, \{3 : 13, 15, 16\}, \{1 : 20, \overline{21}\}, \{5 :, 18, \overline{23}\}, \{6 : 19, 22\}\}$. Each set of internal parentheses denotes a star, beginning with its center label followed by its edge labels. Leaf labels are implied. In the maximal h case, the labelling is 1-1 if and only if each integer $\leq d/2$ appears exactly once (possibly with an overline).

The earlier diagrams plus the following examples show that the bounds in the theorem are sharp:

$$\{(s=6;h=95)\{1:13,14,15,16,17,19\},\{2:18,32,43\},\{3:23,28,41\},$$

$$\{4:21,30,40\},\{5:22,29,39\},\{7:24,27,37\},\{8:25,26,36\},$$

$$\{10:20,31,34\},\{6:44,45\},\{9:33,\overline{42}\},\{11:38,46\},\{12:35,48\}\}$$

$$\{(s=7;h=239)\{1:31,32,33,34,35,36,37\},\{2:38,86,113\},$$

$$\{4:40,83,112\},\{5:41,82,111\},\ldots,\{25:61,62,91\},\{26:39,84,90\},$$

$$\{3:117,119\},\{28:87,\overline{115}\},\{29:89,\overline{118}\},\{30:86,\overline{116}\},\{32:90,117\}\}.$$

2. Introducing Deficits

As implied above, we may write $h = 2n-2l+1-d$ with d a non-negative integer, called the *h-deficit*. Again, $d = 0$ implies that for a union of t stars the integers $\{1,2,\ldots,t\}$ must all label centers. For $d = 1$, h is even which necessitates using $h/2$ to label a center. For $d = 2$, we must have two stars with center labels x, y greater than t. In fact, these two labels must have $x + y = h$ (since any non-center label is necessarily the label of a leaf or pendant edge and the weight at each leaf is h).

By choosing a particular h, we require that all integers less than $h - n$ be used to label star centers. All integers greater than or equal to $h - n$ are naturally paired as $x, \overline{x}$ such that $x + \overline{x} = h$, unless if h is even, in which case $h/2$ has no mate. An even d implies odd h and requires that $d/2$ of the pairs be broken (i.e. assigned to star centers rather than to a leaf and its corresponding pendant edge). For odd d, $h/2$ labels a star center and $(d-1)/2$ pairs are broken. Clearly $d \leq t$ since exactly d stars have labels larger than t.

For a particular h (equivalently, for a fixed d), if the largest star has s leaves, then by considering the minimum possible weight at its center (assuming that $d < t$ so the center may be labelled with 1), we get

$$1 + (t+1-d) + (t+2-d) + \cdots + (t+s-d) \leq 2n - 2l + 1 - d$$

$$s \cdot t + s(s+1)/2 - s \cdot d \leq 2l + 2t - d$$

Using $l < 3t$ and $d < t$ yields the (crude) estimate $t > s(s+1)/14$. This estimate does not take into account the “cost” of breaking pairs. By lowering h, we force labels greater than t be used on some centers. A magic labelling then requires that some smaller labels be available to label that star's edges. This is easiest to accomplish if the larger labels are applied to centers of 2-stars. In the next section, a constraint will be introduced to ensure that enough small labels are available to accomodate the broken pairs.

3. One or more S-Stars Unioned with 2-Stars

It is natural to investigate when large stars can be included within unions. Earlier, we saw that $d = 0$ implied no star could be have more than seven leaves. Moreover, we saw that in order to be vertex magic the union needed to include many more small stars than the minimum that would be necessary to make $l < 3t$.

For a given $s > 3$, we now determine the minimum number of 2-stars needed to form a vertex magic union with an s-star.

We use d to denote the h-deficit so $h = 2n - 2l + 1 - d$. Note that for a union of one s-star and $(t-1)$ 2-stars, $l = s + 2(t-1)$ and $n = 2s + 1 + 5(t-1)$ where l is the number of leaves and n is the number of labels.

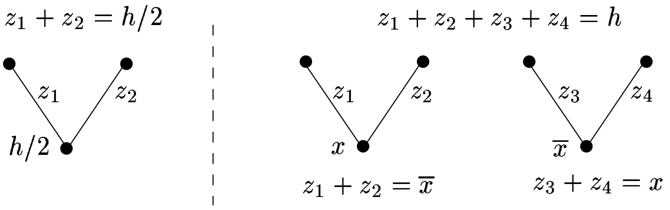

Figure 4. Broken pairs imply large labels on some star centers, requiring that smaller labels be available to label their edges.

Increasing d by two requires breaking a pair. Since that pair of labels will be assigned to centers of 2-stars, this requires a quadruple of labels $\{z_1, z_2, z_3, z_4\}$ such that $z_1 + z_2 = x$ and $z_3 + z_4 = \overline{x}$ where $x + \overline{x} = h$ is the broken pair. Increasing d by one to get an even h requires a pair $\{z_1, z_2\}$ with $z_1 + z_2 = h/2$. These requirements may be combined with the requirement stemming from the minimum weight at the center of the s-star. This gives us:

$$1 + \sigma_{t-d}^{t+s+d} \leq (d/2 + 1)h$$

The "1" on the left hand side corresponds to the minimal label for the center of the s-star. The σ term gives the sum obtained by using the smallest non-center labels available to label the edges of either the s-star or one of the 2-stars with a broken pair center.

This equation gives us a quadratic inequality in d, of the form $d^2 + \alpha d + \beta < 0$, where α and β depend on s and t. Since the coefficient of d^2 is positive, the inequality can only be satisfied if the polynomial $d^2 + \alpha d + \beta = 0$ has real roots. Solving this polynomial gives the following roots:

$$\{t - 7/2 + 1/2 \pm 1/2\sqrt{4t^2 + 20t + 17 - 8st - 4s^2 + 12s}\}$$

Thus, a magic labelling requires $4t^2 + 20t + 17 - 8st - 4s^2 + 12s \geq 0$. Using the quadratic formula, this gives us the necessary condition:

$$t \geq s - 5/2 + \sqrt{2 - 8s + 2s^2}$$

(The other root is negative for $s > 4$ and meaningless.)

Theorem 3.1. *For a union of an s-star with* $t-1$ *2-stars to possess a vertex-magic total labelling, we must have* $t \geq s - 5/2 + \sqrt{2 - 8s + 2s^2}$.

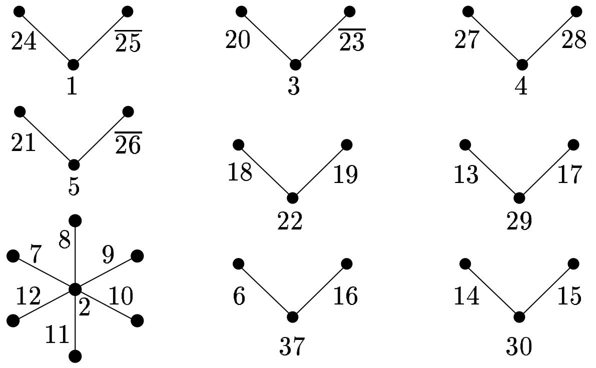

FIGURE 5. This vertex magic graph includes a 6-star and has three fewer stars than possible for the maximal h case. This labelling has $d = 4$ and $h = 39$.

Examples have been found to show that the bound above is strict for $s \leq 13$. As a courtesy to the reader, these labellings will not be listed until the end of the paper.

The above analysis does not explicitly check that the first s available labels are small enough to ensure the weight at the center of the s-star is less than h. Thus a union of stars plus choice of d can satisfy the above inequalities yet not possess a vertex magic total labelling. For example, a "9-star excess" occurs for $s = 9, t = 17$ and $d = 10$. However this graph does have a labelling for $d = 12$. Similarly for $s = 12, t = 24$ and $d = 17$ or $d = 18$, there is a "12-star excess", though $d = 19$ yields a labelling.

An approach similar to the one above can be used to find a necessary condition for a union of m s-stars and $(t - m)$ 2-stars to have a vertex magic total labelling.

Theorem 3.2. *For a union of m s-stars and $(t - m)$ 2-stars to possess a vertex-magic total labelling, we must have*

$$t \geq 1/2 - 3m + ms + \sqrt{2m^2s^2 - 10m^2s + 9m^2 + 2ms - 7m}.$$

4. Unions Including Other Stars

In the previous section, we sought the minimum t required for a union of one s-star and $(t - 1)$ 2-stars to have a vertex magic labelling. However, such a union does not necessarily represent the smallest t for which there exists a vertex magic union of t stars that includes at least one s-star.

In order to investigate the union of an s-star with some combination of 2-stars and 3-stars, we employ an approach similar to the one used above.

Consider the union of one s-star, r 3-stars and $(t - r - 1)$ 2-stars. This gives $l = s + 3r + 2(t - r - 1)$ and $n = 2s + 1 + 7r + 5(t - r - 1)$ Again, we use d to denote the h-deficit.

We now introduce a constraint to guarantee that enough small labels are available to use for the s-star and the 3-stars as well as to cover the cost of breaking pairs.

$$1 + \sigma_{t-d}^{t+d+s+3r} \leq (d/2 + 1 + r) \cdot h$$

Again, this inequality provides a quadratic in d which must possess real roots for there to be valid values for d. We find that the inequality may be satisfied (for some value of d, not necessarily integer) if

$$t \geq -5/2 + s + \sqrt{2 - 8s + 2s^2 - 4r^2 - 16r + 2sr}$$

By taking the derivative of this function with respect to r and setting it to zero, we get $r_{min} = -2 + 1/4s$. This suggests that for larger s, we might find vertex-magic unions involving fewer stars than the bound in the above section. However, the addition of 3-stars introduces additional requirements not included in the above constraint. These will be discussed below. It is possible that necessary conditions may be obtained via this approach but this is a delicate issue since the variables r and t are necessarily integer and the above analysis implicitly assumes them to be continuous.

Consider a union which consists of t_k k-stars for $2 \leq k \leq s$. By first considering the weight of the centers of the largest stars then working our way down, we arrive at a set of s or $s+1$ necessary conditions (some of which involve the cost of breaking pairs). Let $u_i = \Sigma_{k=i}^{s} t_k$, $T = u_2$ and $l_i = \Sigma_{k=i}^{s} k \cdot t_k$, $L = l_2$. To possess a vertex-magic labelling with magic constant $h = 2N - 2L + 1 - d$, we have the following necessary conditions:

For $4 \leq k \leq s$,

$$\sigma_0^{u_k} + \sigma_{T-d}^{T-d+l_k} \leq u_k \cdot h$$

For even d and $2 \leq k \leq 4$,

$$\sigma_0^{u_k} + \sigma_{T-d}^{T+d+l_k} \leq (d/2 + u_k) \cdot h$$

For odd d and $2 \leq k \leq 4$,

$$\sigma_0^{u_k} + \sigma_{T-d}^{T+d-2+l_k} \leq ((d-1)/2 + u_k) \cdot h$$

For odd d, we also need

$$\sigma_0^{u_2} + \sigma_{T-d}^{T+d+l_k} \leq (d/2 + u_k) \cdot h$$

Conjecture 4.1. *For a given union of stars and choice of* d*, the above conditions are sufficient to ensure a vertex-magic total labelling exists with* $h = 2N - 2L + 1 - d$.

5. More General Graphs

In some ways, a large star represents the most extreme behavior for graphs. The center vertex possesses extremely high valence while every other vertex is a leaf. However, the above analysis suggests that arbitrarily large stars may be in included within a vertex-magic graph simply by forming its union with sufficiently many 2-stars. This raises the question – does there exist a graph G such that the union of G with m 2-stars is not vertex-magic for any m?

References

[Wal01] W.D. Wallis, *Magic graphs*, Birkhäuser, Boston, 2001.

s=5	t=6	h=41	{{1:5,6,8,10,11},{2:15,$\overline{17}$},{3:16,$\overline{19}$}, {4:14,$\overline{18}$},{20:9,12},{21:7,13}}
s=7	t=12	h=77	{{4:7,8,9,10,11,12,16},{1:28,$\overline{29}$},{2:36,$\overline{38}$}, {3:27,$\overline{30}$},{5:26,$\overline{31}$},{6:34,37}, {32:22,23},{$\overline{32}$:15,17},{33:19,25}, {$\overline{33}$:13,20},{35:18,24},{$\overline{35}$:14,21}}
s=8	t=14	h=89	{{1:7,8,9,10,12,13,14,15},{2:32,$\overline{34}$},{3:36,$\overline{39}$}, {4:31,$\overline{35}$},{5:33,$\overline{38}$},{6:37,$\overline{43}$}, {40:24,25},{$\overline{40}$:18,22},{41:19,29,},{$\overline{41}$:11,30}, {42:20,27},{$\overline{42}$:16,26},{44:17,28},{$\overline{45}$:21,23}}
s=9	t=17	h=105	{{5:7,8,9,10,11,12,13,14,16},{1:47,$\overline{48}$},{2:49,$\overline{51}$}, {3:50,52},{4:41,$\overline{45}$},{37:33:35},{$\overline{37}$,18,19}, {38:28,39},{$\overline{38}$:15,23},{42:31,32},{$\overline{42}$:17,25}, {43:26,36},{$\overline{43}$:21,22},{44:27,34},{$\overline{44}$:20,24}, {46:29,39},{$\overline{46}$:6,40}}
s=10	t=19	h=118	{{3:7,8,9,10,11,12,13,14,15,16},{1:45,$\overline{46}$}, {2:53,$\overline{55}$},{4:43,$\overline{47}$},{5:44,$\overline{49}$}, {6:51,$\overline{57}$},{59:17,42},{48:29,41},{$\overline{48}$:18,30}, {50:28,48},{$\overline{50}$:19,31},{52:27,39},{$\overline{52}$:20,32}, {54:26,38},{$\overline{54}$:21,33},{56:25,37},{$\overline{56}$:22,34}, {58:24,36},{$\overline{58}$:23,35}}
s=11	t=21	h=127	{{1:4,5,6,7,8,9,10,11,12,13,41},{2:51,$\overline{53}$}, {3:52,$\overline{55}$},{54:23,50},{$\overline{54}$:22,32}, {56:31,40},{$\overline{56}$:14,42},{57:21,49}, {$\overline{57}$:24,33},{58:30,39},{$\overline{58}$:15,43}, {59:20,48},{$\overline{59}$:25,34},{60:29,38}, {$\overline{60}$:16,44},{61:19,47},{$\overline{61}$:26,35}, {62:28,37},{$\overline{62}$:17,45},{63:18,46}, {$\overline{63}$:27,36}}
s=12	t=24	h=146	{{5:6,7,8,9,10,11,12,13,14,15,16,20}, {1:71,$\overline{72}$},{2:61,$\overline{63}$},{3:62,$\overline{65}$},{4:56,$\overline{60}$}, {73:26,47},{55:33,59},{$\overline{55}$:19.36}, {57:35,84},{$\overline{57}$:17,40},{59:34,53}, {$\overline{59}$:18,41},{64:31,51},{$\overline{64}$:22,42}, {66:30,50},{$\overline{66}$:23,43},{67:27,52}, {$\overline{67}$:21,46},{68:29,49},{$\overline{68}$:24,44}, {69:38,39},{$\overline{69}$:32,37},{70:28,48}, {$\overline{70}$:25,45}}

FIGURE 6. These are some labellings which minimize t for a given s-star unioned with 2-stars. The case $s = 6$ is given in an earlier diagram.